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Schwinger-Keldysh propagators from AdS/CFT correspondence

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Abstract: We demonstrate how to compute real-time Green's functions for a class of finite temperature field theories from their AdS gravity duals. In particular, we reproduce the 2×2 Schwinger-Keldysh matrix propagator from a gravity calculation. Our methods should work also for computing higher point lorentzian signature correlators. We elucidate the boundary condition subtleties which hampered previous efforts to build a lorentzian-signature AdS/CFT correspondence. For two-point correlators, our construction is automatically equivalent to the previously formulated prescription for the retarded propagator.

Keywords: Black Holes in String Theory, AdS-CFT and dS-CFT Correspondence, Black Holes.

Contents

1. Introduction

The original AdS/CFT correspondence [1]–[3], motivated by considering a stack of D3-branes in ten dimensional space, states that $\mathcal{N} = 4$ SU(N) super Yang-Mills theory is dual to type IIB string theory in the background of five-dimensional anti-de Sitter space (AdS_5) cross a five-sphere ($\mathbf{S}^5$). Although the string theory in this background is difficult to work with, the low-energy supergravity (SUGRA) limit of string theory is accessible to quantitative calculations. This limit corresponds to the regime of large 't Hooft coupling in the gauge theory. There exists a more general correspondence between conformal field theories at finite temperature and asymptotically AdS spaces containing black holes [4]. Noticing that the real-time formulation of finite temperature field theory (the Schwinger-Keldysh, or close-time-path, formalism) involves a doubling of the degrees of freedom [5]–[7], and that the full Penrose diagram for asymptotically AdS containing a black hole has two boundaries, many have [8]–[10] conjectured that the doubler fields can be thought of as fields living on the second boundary of the AdS dual.

In this paper, we suggest a precise prescription for computing real-time Green's functions from gravity. We show how one can reproduce the full 2×2 matrix of two-point correlation functions for a scalar field and its doubling partner using the AdS dual (recall that in the real-time formalism, the mixed two-point correlators involving one real field and one doubler field do not vanish). Moreover, as our approach is nothing more than a refinement of the usual prescription of taking functional derivatives of a boundary gravitational action, the procedure should easily generalize to higher-point correlators.

There have been many previous attempts to obtain Minkowski-signature correlation functions from AdS/CFT. The early prescriptions for matching correlation functions in the field theory to the classical behavior of bulk fields in the supergravity involved a Wick rotation to a euclidean signature metric. This rotation works only at zero temperature, and perhaps obscures the point that the correspondence should work equally well for the original Minkowski signature. The difficulty with working in the original Minkowski signature is that one generally has greater freedom to set boundary conditions, and it is not clear a priori which boundary condition corresponds to which propagator in a rather large set of Green's functions: advanced, retarded, Feynman, etc.

Although the euclidean correlation functions can in principle be analytically continued to yield the Feynman Green's function, and through it all physical Minkowski signature correlators, it is often desirable to be able to compute the Minkowski signature correlators directly. Direct computation eliminates the need for analytic continuation from a discrete set of Matsubara frequencies which can be technically difficult. Moreover, in finite-temperature AdS/CFT, one can solve the bulk field equation on the gravity side only in the large or small frequency limits. Analytically continuing a function that is only known in certain limits is not always possible.

Minkowski-signature prescriptions have been investigated before [10, 11], and recently a concrete proposal has been put forward [12]. Ref. [12] makes a particular choice of boundary conditions and identifies the retarded Green's function G_R in one particular term of the resulting boundary gravitational action. In (1+1)-dimensional conformal field theory where the euclidean Green's function can be computed exactly and the analytic continuation to a Minkowski signature metric can be performed, the prescription gives the correct G_R [12]. Moreover, in the infrared limit the G_R computed from the prescription satisfies the constraints imposed by hydrodynamics (see e.g. [13]–[15]). However, the prescription of ref. [12] does not follow from taking functional derivatives, and so the prescription cannot be directly generalized to higher point correlators.

In contrast, the prescription presented in this paper allows for the calculation of all correlators, including the higher-point ones, in finite temperature field theories with asymptotically AdS (aAdS) duals. The zero-temperature correlators can be obtained as a limit. We will also see how the doubler fields of the Schwinger-Keldysh formalism emerge from gravity.

Our results come from understanding black hole physics. Our field theories are dual to asymptotically AdS spaces containing black holes, and we draw heavily on the ideas of Hawking and Hartle [16], Unruh [17], and Israel [18] who studied how black holes produce thermal radiation. The choice of boundary condition for the bulk fields in AdS are sensitive to the choice of vacuum and coordinate system in ways that are well studied in the context of Hawking radiation. We will see that the thermal nature of black hole physics gives rise to the thermal nature of the field theory in a more or less straightforward way.

Saving the details for sections 3 and 4, note that the analog of Kruskal coordinates exists for these aAdS spaces containing black holes. The correct prescription for calculating Minkowski signature Green's functions is the usual AdS/CFT prescription worked out by [2, 3] where one selects "natural" boundary conditions at the horizon with respect to the analog of Kruskal time. In the context of Hawking radiation, Kruskal time is often used to define a vacuum state. With respect to the gauge theory time, on the other hand, an observer should see a thermal background. If we were to use gauge theory time to examine the bulk behavior of a field, we would need to take into account this thermal background, a complication which explains some of the early confusion in the literature with respect to these Minkowski signature Green's functions.

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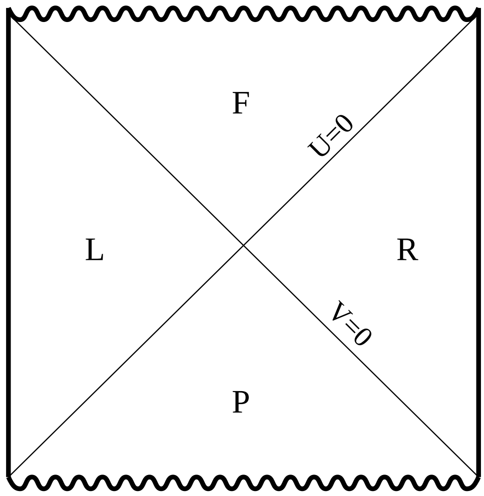

Figure 1: The Penrose diagram for AdS containing a black hole.

In Kruskal coordinates, the full Penrose diagram (see figure 1) for aAdS space becomes apparent. Israel [18] pointed out that the fields in the mirror image universe in the L quadrant of the Penrose diagram should be the doubler fields of the Schwinger-Keldysh formalism in curved space. The authors of [8]–[10] made the further conjecture that the fields on the boundary of the L quadrant should look like ghosts (or doubler fields) from the point of view of the finite temperature CFT dual. Being careful about boundary conditions, we are able to reproduce the 2×2 matrix of propagators for the field and its doubler using the aAdS description.

We begin by reviewing in section 2 the Schwinger-Keldysh formalism for real-time finite temperature field theory. Section 3 contains details of our refined prescription for calculating directly Minkowski signature correlators in AdS/CFT. We focus on the case of a scalar in a non-extremal D3-brane background, but the prescription should be much more broadly applicable. In section 4, we explain how our choice of boundary conditions are related to the boundary conditions imposed by a Feynman propagator.

2. Review of Schwinger-Keldysh formalism for finite-temperature field theory

In the Schwinger-Keldysh formalism, fields (which we denote generically by O) live on a time contour $\mathcal{C}$ which goes from some initial time t_{i} to $t_{\mathrm{i}} - i\beta$, but make an excursion along the real time axis in between. A version of the contour is drawn in figure 2. The contour starts at an initial time t_{i}, goes to some (final) time t_{f}, then turns to the euclidean domain and runs to $t_{\mathrm{f}} - i\sigma$ (where σ is an arbitrary length), after which it runs backward along the real time axis to $t_{\mathrm{i}} - i\sigma$ and then again turns to the euclidean direction and goes to $t_{\mathrm{i}} - i\beta$. The starting point A (corresponding to time t_{i}) and the ending point B (corresponding to t_{f}) of the contour are identified, and one requires that $O|_B = O|_A$ if O is bosonic and $O|_B = -O|_A$ if O is fermionic. For definiteness, we will assume O to be bosonic.

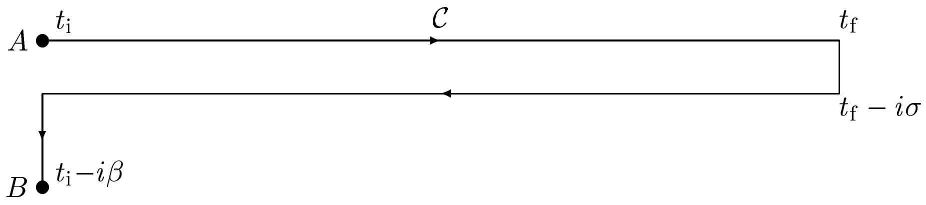

Figure 2: The Schwinger-Keldysh contour.

The parameter σ can be chosen arbitrarily [19]. One possible choice is $\sigma = 0$, in which case the two Minkowski parts of the contour lie on top of each other [5, 7]. To make contact with gravity another choice, $\sigma = \beta/2$, is the most convenient.[1]

The action of a field configuration is the sum of contributions from the four parts of the contour,

$$S = \int_{\mathcal{C}} dt\, L(t) = \int_{t_{\rm i}}^{t_{\rm f}} dt\, L(t) - i\int_0^{\sigma} d\tau\, L(t_{\rm f} - i\tau) - \int_{t_{\rm i}}^{t_{\rm f}} dt\, L(t - i\sigma) - i\int_{\sigma}^{\beta} d\tau\, L(t_{\rm i} - i\tau)\,, \quad (2.1)$$

where

$$L(t) = \int d\vec{x}\, \mathcal{L}[\phi(t,\vec{x})]\,, \quad (2.2)$$

and $\mathcal{L}$ is the lagrangian density.

By introducing a source which is not vanishing on the two Minkowski parts of the contour, one defines the generating functional

$$Z[\phi_1,\phi_2] = \int \mathcal{D}\phi\, \exp\left(iS + i\int_{t_{\rm i}}^{t_{\rm f}} dt \int d\vec{x}\, \phi_1(x) O_1(x) - i\int_{t_{\rm i}}^{t_{\rm f}} dt \int d\vec{x}\, \phi_2(x) O_2(x) \right). \quad (2.3)$$

Here $\phi_{1,2}$ and $O_{1,2}$ are the source and the fields on the two Minkowski parts of the contour, i.e.,

$$\begin{aligned} \phi_1(t,\vec{x}) &= \phi(t,\vec{x})\,, & O_1(t,\vec{x}) &= O(t,\vec{x})\,, \\ \phi_2(t,\vec{x}) &= \phi(t - i\sigma,\vec{x})\,, & O_2(t,\vec{x}) &= O(t - i\sigma,\vec{x})\,. \end{aligned} \quad (2.4)$$

By taking second variations of Z with respect to the source ϕ one finds the Schwinger-Keldysh propagator,

$$iG_{ab}(x-y) = \frac{1}{i^2}\frac{\delta^2 \ln Z[\phi_1,\phi_2]}{\delta\phi_a(x)\,\delta\phi_b(y)} = i\begin{pmatrix} G_{11} & -G_{12} \\ -G_{21} & G_{22} \end{pmatrix}. \quad (2.5)$$

In the operator formalism, the Schwinger-Keldysh propagator corresponds to the contour-ordered correlation function. In contour ordering, time is ordered normally on the upper part of the contour and reversely on the lower part, and moreover any point on the lower part is considered to have larger "contour time" than any point on the upper part. This means

$$\begin{aligned} iG_{11}(t,\vec{x}) &= \langle T O_1(t,\vec{x}) O_1(0)\rangle\,, & iG_{12}(t,\vec{x}) &= \langle O_2(0) O_1(t,\vec{x})\rangle\,, \\ iG_{21}(t,\vec{x}) &= \langle O_2(t,\vec{x}) O_1(0)\rangle\,, & iG_{22}(t,\vec{x}) &= \langle \bar{T} O_2(t,\vec{x}) O_2(0)\rangle\,, \end{aligned} \quad (2.6)$$

[1] The $\sigma = \beta/2$ contour was studied in a field theoretic context in [19, 20].

where $\bar{T}$ denotes reversed time ordering, and

$$\begin{aligned} O_1(t,\vec{x}) &= e^{iHt - i\vec{P}\cdot\vec{x}} O(0) e^{-iHt + i\vec{P}\cdot\vec{x}} \,, \\ O_2(t,\vec{x}) &= e^{iH(t-i\sigma) - i\vec{P}\cdot\vec{x}} O(0) e^{-iH(t-i\sigma) + i\vec{P}\cdot\vec{x}} \,. \end{aligned} \tag{2.7}$$

The Schwinger-Keldysh correlators are related to the retarded and advanced Green's functions, which are defined as

$$\begin{aligned} iG_R(x-y) &= \theta(x^0 - y^0)\langle [O(x),\, O(y)]\rangle \,, \\ iG_A(x-y) &= \theta(y^0 - x^0)\langle [O(y),\, O(x)]\rangle \,. \end{aligned} \tag{2.8}$$

If one goes to momentum space,

$$G(k) = \int dx\, e^{-ik\cdot x} G(x) \,, \tag{2.9}$$

we find that

$$G_A(k) = G_R^*(k) \,, \tag{2.10}$$

and, by inserting the complete set of states into the definitions (2.6) and (2.8), one finds the following relations between the Schwinger-Keldysh correlators and the retarded one,

$$G_{11}(k) = \operatorname{Re} G_R(k) + i \coth\frac{\omega}{2T}\, \operatorname{Im} G_R(k) \,, \qquad \omega \equiv k^0 \,, \tag{2.11}$$

$$G_{12}(k) = \frac{2ie^{-(\beta-\sigma)\omega}}{1 - e^{-\beta\omega}}\, \operatorname{Im} G_R(k) \,, \tag{2.12}$$

$$G_{21}(k) = \frac{2ie^{-\sigma\omega}}{1 - e^{-\beta\omega}}\, \operatorname{Im} G_R(k) \,, \tag{2.13}$$

$$G_{22}(k) = -\operatorname{Re} G_R(k) + i \coth\frac{\omega}{2T}\, \operatorname{Im} G_R(k) \,. \tag{2.14}$$

One sees that when $\sigma = \beta/2$ the matrix G_{ab} is symmetric, $G_{12} = G_{21}$, which makes this choice convenient. We will see that this particular value of σ appears naturally in gravity.

3. Lorentzian field theory correlators from gravity

Our formalism for computing real-time Green's functions should work for a broad class of finite-temperature field theories. The theories are required to have an asymptotically anti-de Sitter (aAdS) space dual with a Schwarzschild-type black hole in the center. The zero-temperature correlators can be obtained in the limit where the black hole vanishes. Typical examples of such asymptotically AdS spaces arise from studying non-extremal D3-branes (AdS_5), M2-branes (AdS_4), or M5-branes (AdS_7). AdS_3 arises in studying collections of non-extremal D1- and D5-branes.

To illustrate the formalism with a concrete example, we will focus on the case of non-extremal D3-branes. We will be using a gravitational description for calculating correlators of $\mathcal{N} = 4$ SU(N) super Yang-Mills theory at finite temperature in the $N \to \infty$ and $g_{YM}^2 N \to \infty$ limit.

The metric on a stack of non-extremal D3-branes is

$$ds^2 = H(r)^{-1/2}\left[-f dt^2 + d\vec{x}^2\right] + H(r)^{1/2}\left(f^{-1}dr^2 + r^2 d\Omega_5^2\right), \tag{3.1}$$

where $H(r) = 1 + R^4/r^4$, $f(r) = 1 - r_0^4/r^4$, $d\Omega_5^2$ is the metric on a unit $\mathbf{S}^5$, and $R^4 \sim N$ is proportional to the number of D3-branes. In these coordinates, there is a horizon at $r = r_0$. In the near horizon limit ($r \ll R$), the metric becomes

$$ds^2 = \frac{(\pi T R)^2}{u}\left(-f(u)dt^2 + d\vec{x}^2\right) + \frac{R^2}{4u^2 f(u)}du^2 + R^2 d\Omega_5^2\,, \tag{3.2}$$

where $T = r_0/\pi R^2$ is the Hawking temperature and we have introduced $u = r_0^2/r^2$. Here, $u = 0$ is the boundary of this aAdS space while $u = 1$ corresponds to the horizon.

As a warm-up and review, let us consider the behavior of a scalar field ϕ of mass m in this aAdS space. The field ϕ obeys the wave equation.

$$\begin{aligned} 0 &= \frac{1}{\sqrt{-g}}\partial_\mu \sqrt{-g} g^{\mu\nu}\partial_\nu \phi - m^2\phi \\ &= 4u^3\partial_u\left(\frac{f}{u}\partial_u\phi\right) + \frac{u}{(\pi T)^2}\left(-\frac{1}{f}\partial_t^2 + \partial_{\vec{x}}^2\right)\phi - m^2R^2\phi\,, \end{aligned}$$

where the $\mu, \nu, \ldots$ run over the indices of the aAdS_5, t, x^1, x^2, x^3, and u. To analyze this differential equation, one makes a Fourier decomposition

$$\phi(x,u) = \int \frac{d^4k}{(2\pi)^4}\, e^{ik\cdot x}\,\phi(k,u)\,. \tag{3.3}$$

The function $\phi(k,u)$ satisfies the equation

$$0 = 4u^3\partial_u\left(\frac{f}{u}\partial_u\phi(k,u)\right) + \frac{u}{(\pi T)^2 f}\left(\omega^2 - f|\vec{k}|^2\right)\phi(k,u) - m^2R^2\phi(k,u) \tag{3.4}$$

which we cannot solve analytically. However, we can analyze the solution at large and small u. At small u, close to the boundary, the space looks like ordinary AdS_5 and we get two possible scalings,

$$\phi(k,u) \approx \phi(k)u^{\alpha_-} + A(k)u^{\alpha_+} \tag{3.5}$$

where $\alpha_\pm$ is a solution to $4\alpha(\alpha - 2) = m^2R^2$. Close to the horizon, $\phi \sim (1-u)^\beta$ where $\beta = \pm i\omega/4\pi T$ and one can choose either sign. In euclidean signature, β is real: $\beta = \pm\omega/4\pi T$ and only one solution is well behaved close to $u = 1$.

The AdS/CFT prescription fixes one boundary condition at the boundary $u = 0$ by fixing $\phi(k)$. In the euclidean case, regularity of the solution at $u = 1$ provides the other boundary condition. However, in Minkowski signature there is an ambiguity, since now both solutions are regular at the horizon.

The authors of ref. [12] take the point of view that the black hole should absorb everything that comes to the event horizon and so take purely incoming boundary conditions. This statement would seem to mean $\phi(x,u) \sim e^{-i\omega t}(1-u)^{-i\omega/4\pi T}$ near the horizon. As we mentioned, the prescription of ref. [12] does not allow one to compute higher-point Green's functions; thus we need a more general prescription. To describe this prescription one needs to use a coordinate system which covers the whole Penrose diagram. This system is analogous to the Kruskal coordinates for Schwarzschild black holes. Indeed, close to the horizon, our metric reduces to a Schwarzschild black hole,

$$ds^2 \to 2(\pi T R)^2 \left(-\left(1 - \frac{2M}{\rho}\right) dt^2 + \left(1 - \frac{2M}{\rho}\right)^{-1} d\rho^2 \right) + \cdots , \tag{3.6}$$

where $T = 1/8\pi M$ and $u = 2M/\rho$. Near the horizon, the transformation to the Kruskal coordinates U and V is the same as for Schwarzschild black holes,

$$\begin{aligned} U &= -4Me^{-(t-r_*)/4M} , \\ V &= 4Me^{(t+r_*)/4M} \end{aligned}$$

where $r_* = \rho + 2M \ln |(\rho/2M) - 1|$, keeping in mind that $\rho \approx 2M$ in this near horizon limit. Kruskal time is defined as $t_K \equiv U + V$ while the radial coordinate can be thought of as $x_K \equiv V - U$.

In these Kruskal coordinates, the structure of the full Penrose diagram for this aAdS space becomes apparent (see figure 1). In the initial discussion of a scalar in aAdS, we were working in the R quadrant where $U < 0$ and $V > 0$ but there are three other quadrants, two of which have singularities. The L quadrant where $U > 0$ and $V < 0$ will be very important in what is to follow. The Kruskal radial coordinate x_K has been chosen so that its value increases as we move from the left to the right of the Penrose diagram.

In what follows we will need to distinguish between incoming and outgoing modes, and positive- and negative-frequency modes. We illustrate the distinctions in the example of four plane-wave solutions with frequency $\pm\omega$, $\omega > 0$,

$$e^{-i\omega U} = e^{-i\omega(t_K - x_K)/2} , \tag{3.7a}$$

$$e^{-i\omega V} = e^{-i\omega(t_K + x_K)/2} , \tag{3.7b}$$

$$e^{i\omega U} = e^{i\omega(t_K - x_K)/2} , \tag{3.7c}$$

$$e^{i\omega V} = e^{i\omega(t_K + x_K)/2} . \tag{3.7d}$$

The modes (3.7a) and (3.7c) are outgoing (the wave front moves to larger x_K as t_K increases), while (3.7b) and (3.7d) are incoming waves. In general, any solution to the plane wave equations can be decomposed into the sum of a function of U and a function of V. The part depending on U is a superposition of (3.7a) and (3.7c) and is outgoing on the R quadrant, while the part depending on V is a superposition of (3.7b) and (3.7d) is incoming on the same quadrant. In our terminology the notion of incoming and outgoing is switched on the L quadrant. We can say that the notion of time t far from the horizon reverses direction in the L quadrant.

The subtleties of defining positive- and negative-frequency modes are well known in the context of quantization of fields on the gravitational background of a black hole [21]. By using t_K to define the vacuum state of the quantum field, one finds that an observer far from the horizon experiences a thermal bath of radiation. This fact can be translated into a horizon boundary condition for our prescription.

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Among the four plane waves considered above, (3.7a) and (3.7b) have positive frequency, and (3.7c) and (3.7d) are of negative frequency. If one extends these mode functions to the complex U and V planes, one sees that the positive-frequency modes (3.7a) and (3.7b) are analytic in the lower half of the U or V planes, while the negative-frequency modes (3.7c) and (3.7d) are analytic in the upper half-planes. Since taking superposition does not alter these analytic properties, one finds that a solution to the wave equation is composed of only positive-frequency modes if it is analytic in the lower U and V half-planes, and vice versa.

For our purposes it will be useful to work in the original t and r coordinates. It is simply a matter of convenience, since any function of t and r can be written in terms of U and V. In the original coordinates one can solve the wave equation separately in the R and L quadrants and obtain one set of mode functions in each quadrant,

$$u_{k,R,\pm} = \begin{cases} e^{ik\cdot x} f_{\pm k}(r) & \text{in R} \\ 0 & \text{in L} \end{cases} \qquad u_{k,L,\pm} = \begin{cases} 0 & \text{in R} \\ e^{ik\cdot x} f_{\pm k}(r) & \text{in L} \end{cases} . \tag{3.8}$$

Some explanation of the notation is needed. In eq. (3.8) k is a four-vector $(k^0, k^1, \ldots)$, so $k\cdot x = -k^0 t + k^1 x^1 + \cdots$. The function f_k is a solution to the scalar wave eq. (3.4) which behaves like $e^{ik^0 r_*}$ near the horizon. We denote $\omega = k^0$. The function $f_k(r)$ has two important properties: $f^*_{-k} = f_k$, and f_k is independent of the sign of $\vec{k} = (k^1, k^2, \ldots)$. By definition, the R modes vanish in the L quadrant and the L modes vanish in the R quadrant. As t and x denote two separate coordinate systems in the R and L quadrants, we add a subscript to distinguish t_L from t_R.

The four modes defined in eq. (3.8), in principle, can be expanded in terms of modes defined in the Kruskal coordinates (3.7). Without performing the explicit Fourier transform, one notices that near the horizon $u_{k,R,+}$ and $u_{k,L,+}$ are functions of U, so they are outgoing modes. Analogously $u_{k,R,-}$ and $u_{k,L,-}$ are functions of V and hence are incoming waves. The modes (3.8) contain, however, both positive- and negative-frequency parts. To separate modes with different signs of frequency one defines, following Unruh [17, 21], the linear combinations that mix modes on the two quadrants,

$$\begin{aligned} u_{1,k} &= u_{k,R,+} + e^{-\omega/2T} u_{k,L,+} , \\ u_{2,k} &= u_{k,R,+} + e^{\omega/2T} u_{k,L,+} , \\ u_{3,k} &= u_{k,R,-} + e^{\omega/2T} u_{k,L,-} , \\ u_{4,k} &= u_{k,R,-} + e^{-\omega/2T} u_{k,L,-} . \end{aligned} \tag{3.9}$$

To ensure that

$$u_1 \sim U^{i\omega/2\pi T} \qquad \text{and} \qquad u_4 \sim V^{-i\omega/2\pi T} \tag{3.10}$$

are continuous across the horizon, a branch cut must be placed in the upper halves of the complex U and V planes. Thus, these modes are analytic in the lower halves of the complex U and V planes, and are positive-frequency. Similarly,

$$u_2 \sim \bar{U}^{i\omega/2\pi T} \qquad \text{and} \qquad u_3 \sim \bar{V}^{-i\omega/2\pi T} \tag{3.11}$$

are analytic in the upper halves of the complex U and V planes and carry negative frequencies. The characteristics of the modes $u_{i,k}$ can be summarized as follows:

$$
\begin{aligned}
u_{1,k}: &\quad \text{outgoing, positive-frequency}\,,\\
u_{2,k}: &\quad \text{outgoing, negative-frequency}\,,\\
u_{3,k}: &\quad \text{incoming, negative-frequency}\,,\\
u_{4,k}: &\quad \text{incoming, positive-frequency}\,.
\end{aligned}
$$

According to the AdS/CFT philosophy, the generating functional $Z[\phi_1, \phi_2]$ can be found by evaluating the action of a solution to the field equations, with a boundary condition such that $\phi_{1,2}$ are the values of the field at the boundaries. The Penrose diagram has two boundaries, so we assume that our field ϕ is equal to ϕ_1 on the boundary of the R quadrant and ϕ_2 on the boundary of the L quadrant. Since a general solution is a superposition of four modes (3.9), one needs to impose boundary conditions at the horizon to eliminate two of the four modes.

Since our goal is to reproduce the Schwinger-Keldysh propagator, which is defined with contour time ordering, it is natural to impose the condition that positive frequency modes should be purely ingoing at the horizon in the R quadrant while negative frequency modes should be purely outgoing at the horizon in the R quadrant. Indeed, in field theory the Feynman propagator $G_F(x-y)$ contains only positive-frequency modes in the limit $x^0 \to \infty$ and negative-frequency modes in the opposite limit $x^0 \to -\infty$.[2] In the next section we will show that at zero temperature one can arrive at this "natural" boundary condition independently by a Wick rotation from euclidean space. These boundary conditions select out u_2 and u_4 as the only components that we can use to describe the bulk behavior of a real scalar field, so

$$\phi(x,r) = \sum_k \alpha_k u_{2,k} + \beta_k u_{4,k}\,. \tag{3.12}$$

We now have enough boundary conditions to specify uniquely the behavior of the scalar field. By requiring that (3.12) approaches $\phi_{1,2}$ on the two boundaries, we can solve for α_k and β_k. The result reads

$$
\begin{aligned}
\phi(k,r)|_R &= \left((n+1)f_k^*(r_R) - n f_k(r_R)\right)\phi_1(k) + \\
&\quad + \sqrt{n(n+1)}\left(f_k(r_R) - f_k^*(r_R)\right)\phi_2(k)\,, \qquad (3.13)\\
\phi(k,r)|_L &= \sqrt{n(n+1)}\left(f_k^*(r_L) - f_k(r_L)\right)\phi_1(k) + \\
&\quad + \left((n+1)f_k(r_L) - n f_k^*(r_L)\right)\phi_2(k)\,, \qquad (3.14)
\end{aligned}
$$

where $n \equiv (\exp(\omega/T) - 1)^{-1}$. The $f_k(r)$ are normalized such that $f_k(r_B) = 1$ at the boundary. In the above expressions, we took the Fourier transform of $\phi(x,r)$ with respect to x_R for the portion in the R quadrant while we took the Fourier transform with respect to x_L for $\phi(x,r)$ in the L quadrant.

[2]For the free propagator this comes from the fact that the poles of G_F as a function of the complex frequency ω are located below the real axis for $\omega > 0$ and above for $\omega < 0$.

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From these two equations (3.13) and (3.14), it is straightforward to read off the bulk-to-boundary propagators. The added complication is that with two source terms and two different space-time (or bulk) regions, there are now four different propagators. For instance, the first term on the right hand side of eq. (3.13) is the bulk-to-boundary propagator for the R boundary and the R bulk. The second term in this equation is the bulk-to-boundary propagator for the L boundary and the R bulk. Eq. (3.14) then gives the bulk-to-boundary propagators for the L bulk.

Now we are finally ready to apply the standard recipe from AdS/CFT correspondence for computing Green's functions. The classical boundary action is

$$\frac{K}{2}\int_R \sqrt{-g}g^{rr}\phi(-k,r)\partial_r\phi(k,r)\frac{d^4k}{(2\pi)^4} - \frac{K}{2}\int_L \sqrt{-g}g^{rr}\phi(-k,r)\partial_r\phi(k,r)\frac{d^4k}{(2\pi)^4}, \tag{3.15}$$

where K is some overall normalization. The conjecture of ref. [12] is that the retarded and advanced Green's functions are related to f_k in the following way,

$$G_R(k) = -K\sqrt{-g}g^{rr}f_k(r)\partial_r f_k^*(r)|_{r_B}\,; \qquad G_A(k) = -K\sqrt{-g}g^{rr}f_k^*(r)\partial_r f_k(r)|_{r_B}\,. \tag{3.16}$$

Using the normalization of the f_k, the radial derivative of $\phi(k,r)$ evaluated close to the R or L boundary is then

$$\begin{aligned} -K\sqrt{-g}g^{rr}\partial_r\phi|_R &= [(1+n)G_R - nG_A]\phi_1 + \sqrt{n(1+n)}(G_A - G_R)\phi_2\,, \\ -K\sqrt{-g}g^{rr}\partial_r\phi|_L &= [(1+n)G_A - nG_R]\phi_2 + \sqrt{n(1+n)}(G_R - G_A)\phi_1\,. \end{aligned} \tag{3.17}$$

The boundary action becomes

$$\begin{aligned} S = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4}\Big[&\phi_1(-k)\left((1+n)G_R(k) - nG_A(k)\right)\phi_1(k) - \\ &- \phi_2(-k)\left((1+n)G_A(k) - nG_R(k)\right)\phi_2(k) + \\ &+ \phi_1(-k)\sqrt{n(1+n)}(G_A(k) - G_R(k))\phi_2(k) + \\ &+ \phi_2(-k)\sqrt{n(1+n)}(G_A(k) - G_R(k))\phi_1(k)\Big]\,. \end{aligned} \tag{3.18}$$

Taking functional derivatives of S with respect to $\phi_1(k)$ and $\phi_2(k)$ yields precisely the Schwinger-Keldysh propagators (2.11)–(2.14) with $\sigma = \beta/2$. One thus concludes that if the conjecture (3.16) is valid, then the Schwinger-Keldysh correlators can be found by taking functional derivatives of the classical action. Vice versa, if one take the classical action as the starting point, then by taking functional derivatives one can find all Schwinger-Keldysh correlators which, in conjunction with eqs. (2.11)–(2.14) will give us the same retarded and advanced Green's functions as computed from the old prescription of ref. [12].[3]

In order to obtain the Schwinger-Keldysh propagator with $\sigma \neq \beta/2$, one can substitute the source $\phi_2(k)$ in the boundary action with $e^{(\sigma-\beta/2)\omega}\phi_2(k)$. The interpretation of this rescaling of $\phi_2(k)$ is not completely clear from the gravity point of view.

[3] For more complicated, composite operators, such as the stress-energy tensor, we expect there may be additional complications arising from contact terms. In particular, we believe that our boundary conditions will continue to produce the Schwinger-Keldysh propagators. However, the relation between G_R and f_k in eq. (3.16) and the relation between G_R and G_{ij} in eqs. (2.11)–(2.14) may change by contact terms.

4. Natural boundary conditions at zero temperature

In zero-temperature field theory one can perform a Wick rotation from euclidean space to Minkowski space. We now show that the "natural" boundary condition at the horizon proposed in the previous section is consistent with this Wick rotation. Consider the zero temperature limit of aAdS space, where we recover a scalar traveling in the Poincare patch of pure AdS_5. In euclidean signature, the behavior of the scalar field is described by

$$f_k^E(z) = \frac{z^2 K_\nu(kz)}{\epsilon^2 K_\nu(k\epsilon)}\,, \tag{4.1}$$

where $\nu = \sqrt{4 + m^2R^2}$, $k = \sqrt{\omega^2 + (k^1)^2 + \cdots}$, and $z = 0$ corresponds to the boundary of AdS_5.[4] The analytic continuation of the Bessel type function K_ν to lorentzian signature is the first Hankel function $H_\nu^{(1)}$ and one finds that in lorentzian signature

$$f_k(z) = \frac{z^2 H_\nu^{(1)}(qz)}{\epsilon^2 H_\nu^{(1)}(q\epsilon)}\,, \tag{4.2}$$

where $q = \sqrt{\omega^2 - (k^1)^2 - \cdots}$. When multiplied by $e^{-i\omega t}$, for large z, $f_k(z)$ corresponds to a wave traveling away from the boundary for $\omega > 0$ while for $\omega < 0$, the wave travels toward the boundary. These boundary conditions are precisely the zero temperature limit of our "natural" boundary conditions.

The boundary conditions on the Feynman propagator are the same conditions that result from an analytic continuation of the euclidean Green's function. This zero temperature calculation tells us that at least at zero temperature, the right boundary conditions are purely outgoing from the boundary for positive frequency and purely ingoing at the boundary for negative frequency modes.

Generalizing now to finite temperature, we are asserting that the boundary conditions should remain the same but we should use the Kruskal coordinates to define them and not the original gauge theory time t.

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$$ds^2 = \frac{(\pm dt^2 + d\vec{x}^2 + dz^2)}{z^2}\,.$$

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